The development of three-dimensional disturbances in an unstable film of liquid flowing down an inclined plane

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On the basis of results from a previous paper, expressions are found for the phase velocity and amplification rate of a wave travelling obliquely to the direction of flow. This wave comprises the general harmonic component of three-dimensional small disturbances, and accordingly a double Fourier integral is introduced to represent a bounded disturbance whose initial distribution over the free surface may be arbitrarily prescribed. Hence an asymptotic approximation is derived for a disturbance which is initially concentrated around a point on the free surface. Several distinctive properties of a localized unstable disturbance are noted: for instance, it lies mainly within an elliptical region whose area increases linearly with time as it moves downstream and which is modulated by long-crested waves. An experimental observation of a growing disturbance on an unstable film is recorded, and its main features are seen to be in agreement with the theory.

In so far as linearized perturbation theory remains applicable, the effects investigated are common to a wide class of parallel and nearly parallel laminar flows. In the final part of the paper the method used to analyse the instability of a film is generalized in order to reveal the connexion between this and other problems; this aim is achieved by demonstrating collective properties of the complete class of flows in question, but particular reference is made to the example of laminary boundary layers and Poiseuille flow between parallel planes.

1. Introduction

In an earlier paper concerning the stability of laminar flow down an inclined plane (Brooke Benjamin 1957; hereafter this paper will be referred to as I), expressions were obtained for the velocity and rate of growth of a small disturbance in the form of an infinitely long-crested sinusoidal wave travelling in the direction of flow. The main purpose of the present paper is to show how these results can be adapted to predict the properties of a wave directed obliquely to the flow and hence, by Fourier integration, can be used to analyse the development of an initially concentrated disturbance. While not adding to the essential solution of the stability problem as established in I, this analysis is useful in providing an interpretation of what actually happens when a film becomes unstable. A photograph is presented in §3 which shows a patch of waves on an unstable film, and the theory appears to account very well for the outstanding features of this disturbance.

The investigation in I used the same basic method that occupies most of the vast literature on the theory of hydrodynamic stability, its essentials being that a two-dimensional wave is assumed to disturb the two-dimensional primary flow and that when a stability criterion for such a disturbance is derived from the linearized equations of motion, it is tested over all values of the wavelength. Justification for considering only two-dimensional disturbances† is given by the theorem due to Squire (1933), which shows a three-dimensional wave disturbance to have the same properties as a two-dimensional one for a smaller flow velocity, i.e. for a smaller Reynolds number; therefore, if the flow is stable until the velocity exceeds a certain value, the neutral wave which marks the limit of stability at this critical value must be two-dimensional. Although Squire's theorem leaves no doubt as to the adequacy of two-dimensional theories in proving theoretical limits of stability, there remains the important yet sometimes neglected point that such theories are generally insufficient to account for the actual event of instability when these limits are exceeded. For unless special measures are taken to introduce disturbances which are approximately two-dimensional, the first manifestation of instability will be of an essentially three-dimensional character since initial disturbances will arrive at the unstable region of the flow more or less randomly in spatial distribution as well as in time.

Under slightly supercritical conditions in general, linearized theory indicates a range of amplified waves both two- and three-dimensional, yet with a varying rate of amplification which is greatest for a certain two-dimensional wave; and so, during the time the theory remains applicable, this wave will tend to become the predominant component in the Fourier analysis of a developing transient disturbance. This property accounts for the occasional successes of two-dimensional theories in predicting the 'most prominent' wave in observed cases of instability developing naturally (i.e. without artificial excitation); for instance, waves in keeping with the Tollmien–Schlichting theory were observed in the experiments of Schaubauer & Skramstad (see Schlichting 1955, ch. 16, §*e*) when the disturbances in a laminar boundary layer were allowed to arise spontaneously from background turbulence which had been reduced to the exceptionally low level of 0·03 %; and also in this respect the theory in I was found to check with experimental observations made by Binnie (1957). It is seldom the case, however, that initial disturbances are so small as to allow enormous amplifications following linearized theory, to the extent that the aforementioned selective process leaves the manifestation of instability with little trace of its original three-dimensional character. Linearized theory may nevertheless often account for a preliminary stage of amplification, during which the optimum long-crested wave may develop some degree of prominence yet other components still remain significant, particularly those three-dimensional ones neighbouring to the optimum.

These considerations rather suggest that the natural role of linearized perturbation theories in relation to many problems of hydrodynamic stability—particularly ones where non-linear effects are known to arise readily—is to de-

† More precisely, it is justification for considering only the components with $\beta = 0$ in a Fourier analysis of an arbitrary small disturbance in terms of wave-numbers α, β respective to the direction of flow and the transverse direction.

scribe the growth of localized essentially three-dimensional disturbances. As the basic element composing the preliminary stages of instability, one may usefully imagine an expanding region of perturbed flow which is initiated from a fairly tightly concentrated disturbance swept into the unstable part of the stream; as the region grows, the optimum long-crested wave will become progressively more prominent within it. To reiterate the main point of the present argument, we suggest there may often be a phase of significant duration described by linearized theory (even long enough to permit asymptotic approximations—as introduced in the following analysis) but in which the perturbed region does not expand so far as to cease to be an essentially three-dimensional entity—say by reaching all lateral boundaries of the flow. The over-all structure of the unstable flow, at least of its preliminary stages, may be conceived as an assembly of such elements; where they overlap, the optimum wave components need not coincide in phase, and it seems possible that phase randomness among the predominant spectral component often contributes largely to the disorder observed in hot-wire signals detecting the onset of boundary-layer instability.

The present problem may be proposed as a helpful illustration of this general aspect of stability theory, both because the properties of three-dimensional transient disturbances in a film can be found approximately without much difficulty, and also because the effects in question can readily be observed experimentally. In both these respects the problem is unusually straightforward; it is, for instance, much more so than the corresponding problem for an unstable boundary layer. Apparently no complete theory of the growth of small three dimensional disturbances in any flow of boundary-layer type has yet been given in the literature,† though there have been several investigations of the properties of individual three-dimensional waves (see particularly Watson 1960). Nor apparently have any directly relevant experimental observations been reported. Accordingly, one object of this paper is to emphasize the illustrative value of the film problem in this general connexion. The method of analysis applied in §2 to the main problem will be generalized in §4 in order to show how the properties exemplified by a film may prevail in other circumstances; and in particular it will be shown how, proceeding from the corresponding theory of two-dimensional disturbances, a calculation like that in §2 can be completed for plane Poiseuille flow and for laminary boundary layers.

One feature to be demonstrated in §4 is in fact absent from the main problem, which is thereby made specially simple. This is the dispersive effect arising from variations in the velocity of propagation of unstable waves. The velocity is constant in the approximate theory applied to the problem of an unstable film.

2. Analysis

The problem to be considered is indicated in figure 1. Liquid with density ρ, viscosity $\rho\nu$ and surface tension $\rho\Gamma$ flows in a uniform film of thickness h down a

† But Dr W. Criminale presented a paper on this subject at the AGARD conference on boundary-layer theory in London earlier this year (1960). He has made extensive calculations for a particular velocity profile, so dealing comprehensively with an example of the general problem covered briefly in §4.

plane inclined at an angle θ to the horizontal. This primary flow is steady and laminar, so that the velocity profile is a parabolic arc with its vertex in the free surface. The velocity u_0 of the free surface will be taken as the unit of velocity for the purpose of defining dimensionless variables, and h will be taken as the unit of length. As shown in figure 1, Cartesian axes (x, y, z) are taken with x directed down the plane parallel to the flow and z directed transversely. Our main object is to find the properties of a small disturbance superposed on this flow and initially concentrated around a point in the (x, z)-plane; subsequently, if the flow is unstable, the disturbance will spread over a progressively wider area of the plane.

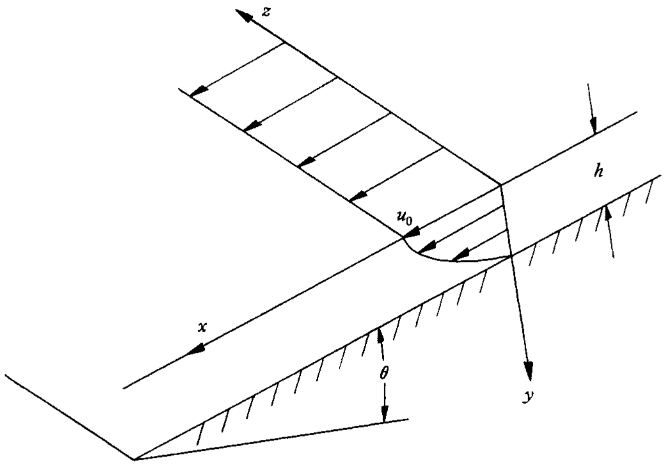

FIGURE 1. Diagram of the undisturbed flow, showing the velocity profile.

The analysis will be based on the following results which were obtained in I for the case of disturbances independent of z. The displacement of the free surface from its equilibrium plane was represented by the real part of

$$\eta = \delta e^{i\alpha(x-ct)}, \tag{1}$$

where δ is a constant representing the initial wave amplitude. There is no need here to express the corresponding velocity perturbations; it is sufficient to recall that they depend on x and time t in the same way as η, and that they are functions of y which vanish at the solid boundary. The waves observed in practice when the conditions of stability are slightly exceeded are extremely long compared with the film thickness; accordingly, an approximate theory was developed in I, §5, on the assumption that the dimensionless wave-number α ($= 2\pi h/$wave-length) is small. On this basis, the real and imaginary parts of c were found to be

$$c_r = 2, \tag{2}$$

$$c_i = \tfrac{1}{3}R\{(\tfrac{8}{5} - G)\alpha - T\alpha^3\}, \tag{3}$$

where $R = u_0 h/\nu$ is the Reynolds number, and where $G = gh\cos\theta/u_0^2$ and $T = \Gamma/hu_0^2$ are numbers characterizing the effects of, respectively, gravity and surface tension on the disturbance. Note that c_r and c_i as given by (2) and (3) express the respective physical quantities as multiples of u_0. The parameters R

and G are not independent in this problem, since the primary flow itself constitutes a balance between viscous forces and the weight of the liquid (see I, §2): their relationship is in fact

$$G = \frac{2\cot\theta}{R} = \frac{8R_c}{5R}, \tag{4}$$

in which R_c is the critical value of R defined in the next paragraph. (The notation of (3) differs somewhat from the original presentation in I; in particular, the Reynolds number in I was based on the mean velocity and so is equivalent to $\frac{2}{3}R$ here.)

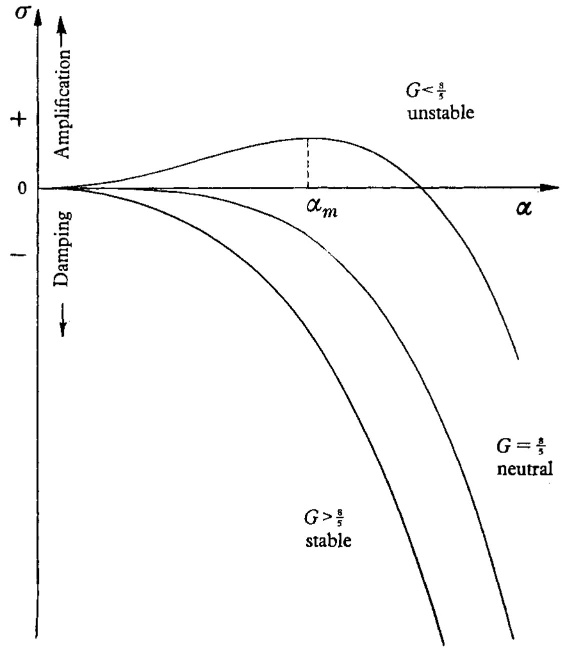

FIGURE 2. Typical graphs of amplification rate $\sigma = \alpha c_i$ as a function of wave-number α according to equation (3). Here σ is proportional to $k\alpha^2 - \alpha^4$, where k is a number which is positive, zero and negative, respectively, in the unstable, neutral and stable cases.

The sign of $\sigma = \alpha c_i$ is a criterion of stability since σt forms the real part of the exponent on the right-hand side of equation (1). Thus equation (3) shows that the flow is unstable when $G < \frac{8}{5}$ [i.e. $R > R_c$, where $R_c = \frac{5}{4}\cot\theta$], because then $\sigma > 0$ for a range of small values of α, that is, there are waves whose amplitude increases indefinitely with time. Some typical graphs of σ *vs* α are drawn in figure 2 in order to show clearly the meaning of this result. Under unstable conditions, σ has a positive maximum σ_m (i.e. there is a maximum rate of amplification) at a wave-number given by

$$\alpha_m^2 = \frac{\frac{8}{5} - G}{2T} = \frac{4(1-[R_c/R])}{5T}; \tag{5}$$

and the fact that α_m is indefinitely small when the stability condition is just exceeded confirms the appropriateness of the small-α approximation. When the plane is vertical, we have $G = 0$ for every R and hence we see that the flow is always unstable. However, it was shown in I that for a vertical film σ_m is extremely small for Reynolds numbers below a certain 'quasi-critical' range, but in this range it suddenly becomes large. This explains the sudden appearance of waves in experiments where the Reynolds number is gradually increased from small values. For water at normal temperatures the quasi-critical range is near $R = 6$, and the corresponding α_m is still quite small. Experiments made by Binnie (1957, 1959) have substantially confirmed the stability theory summarized here.

[It is noted incidentally that these results also apply to a film running down the under side of an inclinded plane. As $\frac{1}{2}\pi < \theta < \pi$ in this case, so that $G < 0$, equation (3) shows gravity to have a destabilizing effect, as would be expected on physical grounds. The film is always unstable for $\frac{1}{2}\pi \leqslant \theta \leqslant \pi$, and in the limit $\theta \to \pi$ which makes $u_0 \to 0$ the present results describe the Taylor instability of a stationary liquid layer (cf. I, p. 568).]

Let us next consider a three-dimensional disturbance for which the displacement of the free surface may be represented by

$$\eta = \delta \exp\{i(\alpha x + \beta z) - i\alpha c t\}. \tag{6}$$

Allowing complex values of δ and both positive and negative values of α and β, we see that this expression constitutes the most general Fourier component in the (x, z)-plane; that is, every possible simple-harmonic function of x and z is obtainable by linear superposition of the real parts of such expressions. At the same time we recognize that the disturbance represented is essentially a long-crested wave having a wave-number $\gamma = (\alpha^2 + \beta^2)^{\frac{1}{2}}$ and propagating at velocity $(\alpha/\gamma)\, c_r$ in a direction inclined at an angle $\tan^{-1}(\beta/\alpha)$ to the x-axis [i.e. c_r is the phase velocity in the x-direction, which is of course larger than the velocity in the direction of propagation].

Expressions for c_r and c_i in this case can at once be inferred from (2) and (3) by appeal to the well-known theorem due to Squire (1933), and further explained by Yih (1955), which states in effect that the properties of the disturbance depend on the primary flow only in respect of its component in the direction of propagation. Thus (2) is to be interpreted as an expression for the wave velocity in the oblique direction, with $(\alpha/\gamma)\, u_0$ instead of u_0 taken as the unit of velocity; and so, reverting to the original unit, we deduce immediately that the wave velocity in the x-direction is

$$c_r = 2 \tag{7}$$

in this case as before. A similar interpretation holds for (3) with $(\alpha/\gamma)\, u_0$ replacing u_0 in the definitions of R, G and T (e.g. $(\alpha/\gamma)\, R$ must be written in place of R if R is to retain its previous meaning); and hence we get

$$c_i = \frac{1}{3}\frac{\alpha}{\gamma} R\left\{\tfrac{8}{5}\gamma - G\frac{\gamma^3}{\alpha^2} - T\frac{\gamma^5}{\alpha^2}\right\}, \tag{8}$$

where R, G, and T mean the same as before. The logarithmic rate of growth is therefore

$$\sigma = \alpha c_i = \tfrac{1}{3} R\{\tfrac{8}{5}\alpha^2 - G(\alpha^2 + \beta^2) - T(\alpha^2 + \beta^2)^2\}. \tag{9}$$

Figure 3 shows a typical contour map of σ in the (α, β)-plane. The maxima of σ occur at points $\pm\alpha_m$ along the α-axis, this optimum value of α being, of course, the value given by (5). The contour $\sigma = 0$ divides the plane into stable and unstable regions, since waves for which the point (α, β) lies outside this contour are damped ($\sigma < 0$) and waves for which (α, β) lies inside are amplified ($\sigma > 0$).

Introducing the moving axis $x' = x - c_r t$, whose origin travels downstream at the constant velocity $c_r = 2$, we now have that the most general harmonic component is

$$\delta \exp\{i(\alpha x' + \beta z) + \sigma(\alpha, \beta)\, t\},$$

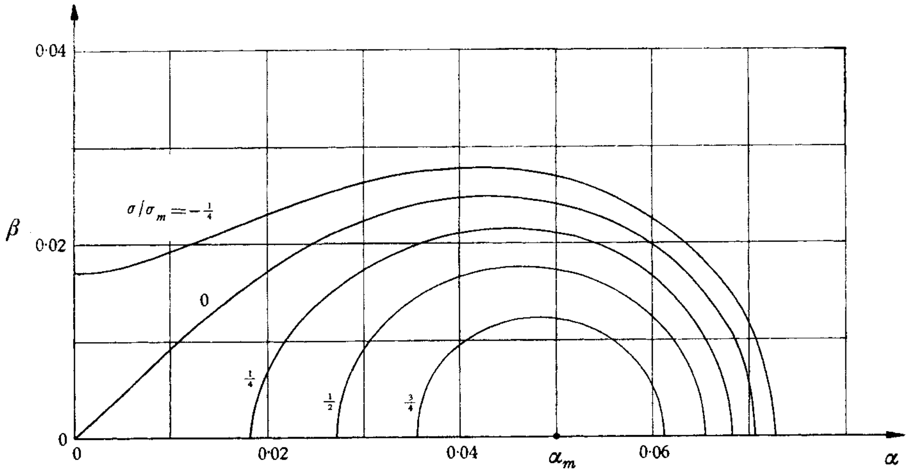

FIGURE 3. Contours of constant amplification rate σ in the first quadrant of the (α, β)-plane; the figure in the complete plane is symmetrical about both axes. In this example it is specified that $G = \frac{4}{5}$ and $T = 160$; hence $\sigma_m = R/3000$ and $\alpha_m = 0{\cdot}05$.

where δ is independent of x', z and t, and where α and β are any pair of positive or negative real numbers. Hence disturbances of arbitrarily prescribed form at $t = 0$ can be represented by double Fourier integrals of the type

$$\eta = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(\alpha, \beta) \exp\{i(\alpha x' + \beta z) + \sigma t\}\, d\alpha\, d\beta. \tag{10}$$

We are here restricted to initial disturbances $\eta(x', z, 0)$ for which there exists a Fourier transform $\delta(\alpha, \beta)$ in the real variables α and β (e.g. we cannot deal with functions, such as periodic ones, which do not vanish at infinite distances from the origin), but all cases relevant to the physical problem under consideration are clearly admissible.

Whatever the initial distribution $\delta(\alpha, \beta)$ of wave components, it is clear that, as t increases, those components for which σ lies very close to the maximum σ_m will eventually predominate over all others; and on this principle an asymptotic approximation to (10) may readily be obtained by Laplace's method (Erdélyi 1956, §2·4). The general result that η eventually depends only on $\delta(\alpha_m, 0)$ will be confirmed in §4, and for the present we shall fix attention on the following example. Taking $\delta(\alpha, \beta) = \text{const.}$ in (10), we have a disturbance which at $t = 0^+$ is highly concentrated in the vicinity of the point $x' = 0$, $z = 0$, and which at this and subsequent times is represented by

$$\eta = \int_0^{\infty}\int_0^{\infty} \cos \alpha x' \cos \beta z\, e^{\sigma t}\, d\alpha\, d\beta, \tag{11}$$

where for simplicity an amplitude factor of unity is taken. While the fact should be kept in mind that the asymptotic approximation to be derived is not dependent on the special choice of δ, this example serves particularly well to demonstrate esssentials. The initially uniform distribution of wave components is the most clearly conceivable starting-point for the process of selective amplification; and there is the important practical aspect that with such a distribution the asymptotic approximation becomes reliable sooner than with a non-uniform initial distribution having its greatest density away from the point of maximum amplification. Thus initially concentrated disturbances are the most likely to manifest the predicted asymptotic behaviour distinctly in practice, a conclusion that was borne out by the experiments described in §3.

Laplace's method takes the leading terms of the expansion of σ about σ_m, which are

$$\sigma = \sigma_m - a(\alpha-\alpha_m)^2 - b\beta^2, \tag{12}$$

with

$$a = \tfrac{2}{3}R(\tfrac{8}{5}-G), \quad b = \tfrac{8}{15}R. \tag{13}$$

The approximation to (11) is therefore

$$\eta \sim e^{\sigma_m t}\int_0^\infty \cos\alpha x'\, e^{-at(\alpha-\alpha_m)^2} d\alpha \int_0^\infty \cos\beta z\, e^{-bt\beta^2} d\beta. \tag{14}$$

In (14) the integral with respect to β is a standard result, and that with respect to α is reducible approximately to a similar integral by changing the lower limit to $-\infty$, the effect of this change being within the over-all error. The outcome is, apart from a numerical factor with no significance here,

$$\eta \sim \frac{1}{t} e^{\sigma_m t} \cos\alpha_m x' \exp\left\{-\frac{x'^2}{4at} - \frac{z^2}{4bt}\right\}. \tag{15}$$

The following features of this result are worth noting separately:

(1) The amplitude of the disturbance grows according to $t^{-1}e^{\sigma_m t}$, not exponentially as might be expected. [Note that for a two-dimensional localized disturbance the corresponding result would be $t^{-\frac{1}{2}}e^{\sigma_m t}$; in the present case each of the two integrals in (14) contributes a factor $t^{-\frac{1}{2}}$.]

(2) The disturbance remains centred at the origin of x'; thus the developing wave-pattern is transported downstream at the velocity $c_r = 2$.

(3) The disturbance is effectively confined within an elliptical region whose dimensions grow proportionally to $t^{\frac{1}{2}}$, i.e. its area increases linearly with time.

(4) The ratio of the axis of the ellipse in the flow direction to the transverse axis is

$$\epsilon = \left(\frac{a}{b}\right)^{\frac{1}{2}} = (2-\tfrac{5}{4}G)^{\frac{1}{2}} = 2^{\frac{1}{2}}\left(1-\frac{R_c}{R}\right)^{\frac{1}{2}}. \tag{16}$$

The ellipse is therefore very oblong in the transverse direction when the film is just unstable, i.e. when R slightly exceeds R_c. The outline of the disturbance becomes circular ($\epsilon = 1$) when $R = 2R_c$; and when R is more than twice its critical value, the disturbance becomes elongated in the flow direction. For a vertical film, $G = 0$ and so $\epsilon = \sqrt{2}$.

[Equation (16) also applies to a film on the under side of an inclined plane (i.e. $\frac{1}{2}\pi < \theta < \pi$, $G < 0$ and hence $\epsilon > \sqrt{2}$), provided the inclination is not too far from the vertical. When $\theta \to \pi$, (16) indicates that $\epsilon \to \infty$; but this result is spurious, as readily appears on reconsideration of the expression (9) for σ. For, if the film lies beneath a nearly horizontal plane, so that $-G$ is large yet $G/T = gh^2 \cos\theta/\Gamma \to -gh^2/\Gamma$, then the optimum wave is determined primarily by the second and third terms on the right-hand side of (9). Hence σ is very nearly equal to σ_m for $\alpha^2+\beta^2 = gh^2/2\Gamma$, i.e. on a circle centred at the origin in the (α, β)-plane; and though, if the plane is not quite horizontal, absolute maxima σ_m still occur at $\alpha = \pm\alpha_m$, $\beta = 0$, they are not distinct enough for the present asymptotic approximation to be reliable.]

Finally, as regards physical applications of the foregoing analysis, the following points need mention:

(*a*) The expressions that were used for c_r and c_i are accurate only when α and β are small, yet the initial disturbance was represented as a synthesis of an indefinitely wide range of wave-numbers over which these expressions were taken to hold. This step is legitimate, however, as a convenient means to obtaining the asymptotic approximation, i.e. our description of the disturbance when it has grown large enough to be composed from the optimum part of the wave-number spectrum. (Our assumption that α is small in the optimum range relies, of course, on (5).) Components at large wave-numbers are known to be rapidly damped anyway (see I, §4), and so it is immaterial that they are assigned an incorrect though still rapid rate of damping.

(*b*) Regarding the accuracy of (15) as an asymptotic approximation, a statement as to a sufficient magnitude of t is still lacking. By developing the asymptotic expansion of which (15) is the first term, one can see that (15) would certainly be a close approximation whenever t is large enough to make $\sigma_m t/\alpha_m$ substantially greater than the major axis of the ellipse. When t is this large, (15) becomes accurately applicable whatever the form of the initial disturbance; but, at least for the case of a concentrated initial disturbance as represented by (11), it would appear that (15) still describes roughly the main features of the wave-pattern when t is much smaller than this. There seems little point in attempting to improve the present approximation on the basis of the simple formulae taken from I, because the limitations of these formulae in respect of the matter (*a*) above is probably just as important as the limitations of (15) as an approximation to (11) or, as it is more generally (cf. §4), an approximation to (10).

(*c*) A more serious consideration than either of the above is that before the asymptotic behaviour according to linearized theory is approximately established, a disturbance may grow to a size at which non-linear effects become important. The initial disturbance clearly must be quite small for the present results to be applicable at some subsequent stage, but it would be very difficult to predict the practical limitations in this regard, and the matter is best left to be settled by experiment.

3. An experimental observation

Figure 4 (plate 1) shows an example of the phenomenon in question, and this photograph happily bears out the approximate theory proposed above. It is thought desirable to include this isolated observation for its illustrative value, even though a comprehensive experimental check on the predictions listed below equation (15) has not yet been made. The experiment that provided figure 4 was done in collaboration with Mr W. Troutman.

In the experimental arrangement a thin film was formed on an inclined sheet of plate glass by allowing water to flow through a gap under a rectangular bar spanning the sheet. The pool of water above the gap was supplied by seepage through layers of muslin, so that the pool was very little disturbed by the incoming flow; and by straining the glass sheet careful adjustments of the gap were made in order to produce a film that was approximately uniform over a fairly wide span. The velocity u_0 of the free surface was measured by timing the progress of dust particles in the surface, and then the film thickness h was calculated by means of the following formula given by rearrangement of (4):

$$h^2 = 2u_0\nu/g\sin\theta. \tag{17}$$

Hence the Reynolds number $R = u_0h/\nu$ was obtained. This estimate of R was found generally to agree closely with the estimate $R = 3Q/2\nu$ in which Q, the volume flow rate per unit span, was measured by collecting and weighing the flow.

The long-crested waves that can be seen near the bottom of figure 4 (plate 1) developed spontaneously, presumably as the outcome of small fluctuations in the stream which in turn were due to disturbances in the pool of water behind the gap or to vibrations picked up by the whole apparatus. The regularity of these waves can be attributed to the fact that any disturbance was rapidly transmitted over the surface of the pool and so communicated to the whole span of the film.

At a good distance above the place where the naturally excited waves first appeared, localized disturbances were introduced into the film either by touching the free surface momentarily with a fine wire or by delivering it a minute puff of air through a capillary tube; these methods were equally effective, though the first was more easily controlled. The disturbances were made at points about 2 cm downstream from the opening of the gap; it was unsuitable to make them any nearer than this to the beginning of the film since a disturbance was then liable to be spread unduly by being transmitted along the meniscus attached to the vertical face of the bar. A sequence of very short ripples was seen to radiate rapidly from the initial disturbance, but these ripples were damped out within a short distance. If, however, the film was unstable, a pattern of much longer waves was also seen to develop from the disturbance, increasing progressively in area and amplitude as it travelled downstream.

When the present photograph was taken, the slope θ was $13\cdot1°$ and u_0 was found to be 6·35 cm/sec. The temperature of the water was 18·6° C, and hence it was estimated that $\nu = 0\cdot0104\,\text{cm}^2/\text{sec}$ and $\Gamma = 73\cdot0\,\text{cm}^3/\text{sec}^2$. With this u_0 and ν, (17) gives $h = 0\cdot0244$ cm, and it follows $R = 14\cdot9$. The critical Reynolds number for this slope is $R_c = \frac{5}{4}\cot\theta = 5\cdot4$, and thus it is confirmed that the film was unstable.

The disturbance shown in figure 4 was photographed when it had travelled about 25 cm downstream from its point of origin. This patch of waves is seen to have an approximately elliptical outline, and the wave crests are seen to be approximately parallel at right angles to the flow direction. The patch is slightly oblong in the flow direction, and ϵ may be estimated to be about 1·1 With the data given in the last paragraph, the theoretical value of ϵ according to (16) is 1·13. From the photograph the optimum wavelength may be estimated as $\lambda_m = 2{\cdot}1$ cm, and the theoretical value found from (5) is $\lambda_m = 1{\cdot}85$ cm, corresponding to $\alpha_m = 0{\cdot}083$; this seems a reasonable agreement (cf. Binnie 1959). It was also confirmed, though only roughly, that the waves travelled at about twice the velocity of the free surface, and that after a disturbance had developed into a distinct pattern such as shown in figure 4, its area grew approximately linearly with time. The latter property was most readily checked by observing that the extremities of the transverse axis described a parabola as the axis moved downstream at constant speed.

The stage at which the theory ceased to apply, owing to the development of significant non-linear effects, was indicated by deformations of the wave-pattern in the following way. Along each transverse ridge formed by a wave crest, the central parts where the amplitude was largest began to travel considerably faster than the outlying parts, so that the ridge became progressively more convex in the downstream direction. However, it could sometimes be observed that even when the waves in the interior of a patch had grown so large as to be greatly distorted by the non-linear effects, the lateral outskirts of the patch continued to spread in the manner predicted by the linearized theory.

4. Generalization of the previous analysis with application to unstable laminar flows of boundary-layer type

An advantage of the problem previously considered is that it affords a simple illustration of properties common to a wide class of parallel or nearly parallel flows, which includes Poiseuille flow between parallel planes and laminar boundary layers. Accordingly, it will now be shown how the mathematical arguments used in §2 may be extended to other examples of hydrodynamic instability possessing the distinctive feature that a certain two-dimensional wave disturbance has a maximum rate of growth for given conditions of flow. The general method of approach will be explained with particular reference to the important cases just mentioned, for which the mechanism of instability depends only on fluid inertia and a constant viscosity; the previous example should suffice to indicate the procedure when there are other physical factors (e.g. gravity and surface tension acting on a free surface, or continuous variations of density or viscosity). The properties of a three-dimensional disturbance in plane Poiseuille flow or in a boundary layer will not, however, be evaluated explicitly as was done for the previous example; the aim here is merely to give a formal demonstration of their general character. In addition to properties already exemplified in §2, we have to consider the effects of variations of c_r with wave-number.

The solution to the stability problem for plane Poiseuille flow is now well established (Lin 1955, ch. 3), and this may be considered the archetype of the various available theoretical results which are adaptable to present treatment. The boundary layer formed by incompressible flow over a flat plate presents a closely related stability problem, having usually been treated as if the primary flow were parallel (Lin 1955, ch. 5; Schlichting 1955, chs. 16 and 17). Various predictions based on this approximation have been confirmed experimentally; but we must note with caution that our use here of the approximate theory, neglecting boundary-layer growth and the change of Reynolds number in the direction of flow, stands in need of special justification. On this basis an analysis of the development of a discrete disturbance as it moves downstream remains secure only while the distance travelled, as well as the length of the disturbance, covers insignificant changes of boundary-layer thickness. Nevertheless, it seems reasonable to suppose that the usefulness of the theory could be considerably extended by taking the boundary-layer parameters to vary slowly in the results worked out on the assumption of fixed parameters.

In the existing analyses of these problems, the equations of motion are linearized in terms of velocity and stress perturbations in the general form

$$v = \hat{v}(y; \alpha, R)\, e^{i\alpha(x-ct)}, \tag{18}$$

where α is real. The 'wave amplitude distributions' $\hat{v}$ are functions only of the co-ordinate y perpendicular to the solid boundaries, i.e. they are indepenent of time t and the co-ordinate x in the flow direction; but they depend parametrically on α and Reynolds number R. For the velocity perturbation parallel to y, the respective function $\hat{v}$ is a solution of the well-known Orr–Sommerfeld equation, and for the velocity parallel to x the respective $\hat{v}$ is the first derivative of a solution (see Lin, ch. 1). There is no need, however, for us to recall the relationships among the velocity and stress components; and, to fix ideas regarding the physical meaning of the subsequent analysis, we can henceforth take $\hat{v}$ to refer specifically to the velocity parallel to y.

Consideration of two-dimensional wave disturbances in the form (18) is sufficient to solve the stability problem, since Squire's theorem proves that such a wave is the one maintained at the 'critical' Reynolds number—i.e. at the limit of stability. With the boundary conditions, the linearized dynamical equations and the continuity equation constitute a characteristic-value problem leading to a relationship of the type (see Lin, ch. 3)

$$F(\alpha, R, c) = 0. \tag{19}$$

Each pair of values α and R thus specifies a value of c, which is generally complex. (In other words, the Orr–Sommerfeld equation and the boundary conditions on its general solution comprise a system which provides *discrete* eigenvalues c and eigenfunctions $\hat{v}(y;\, \alpha, R)$.)

This relationship is extremely complicated, in contrast to the simple case discussed in § 2, and it cannot be solved so as to give c explicitly in terms of general values of α and R. The practical solution to the stability problem is instead usually presented as a 'curve of neutral stability' in the (α, R)-plane, which is worked out numerically by putting $c_i = 0$ and eliminating c_r between the real and imaginary parts of (19) [see Schlichting (1955, pp. 328, 329) for an example]. Further contours of c_i have been worked out for some problems (see figure 5). Though certain approximations can be made, justified by α being fairly small yet αR large, the calculations are still very laborious. Even so, and notwithstanding the form of the numerical results, it is implicit in this procedure that c is a definite function of α and R. That is, we can write

$$c = c_r + ic_i = \mathbf{c}(\alpha, R), \tag{20}$$

using bold type to denote the prescribed function, and we know that this function may at least be estimated by interpolation on the available contour maps of c_r and c_i in the (α, R)-plane. [Incidentally, it is easily verified that $\mathbf{c}(-\alpha, R)$ is the complex conjugate of $\mathbf{c}(\alpha, R)$, which we denote by $\tilde{\mathbf{c}}(\alpha, R)$.]

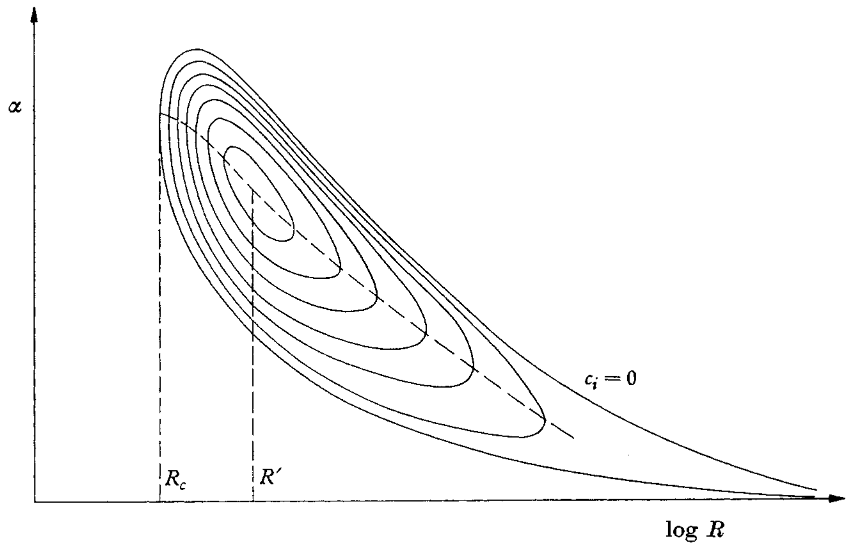

FIGURE 5. Contours of constant c_i (zero and positive values) in the (α, R)-plane, with R expressed on logarithmic scale. The dashed curve is the locus of the maxima of c_i for given values of R, and an absolute maximum occurs at $R = R'$; this curve necessarily lies below the locus of the maxima of σ.

As the key to the present application of stability theory, Squire's theorem may be restated as follows. For a perturbation in the form

$$v = \hat{v}(y) \exp\{i(\alpha x + \beta z) - i\alpha ct\}, \tag{21}$$

where z is the co-ordinate perpendicular to (x, y), the result corresponding to (20) is

$$c = \mathbf{c}(\gamma, R^*), \tag{22}$$

where $\gamma = (\alpha^2 + \beta^2)^{\frac{1}{2}}$ and $R^* = (\alpha/\gamma) R$ [this notation follows Watson (1960)], and where $\mathbf{c}$ is the *same* function of the two variables that occurs in the solution (20) to the two-dimensional problem. (Note that γ takes the same sign as α.) Furthermore, the amplitude factor in (21) satisfies a form of the Orr–Sommerfeld equation with parameters γ, R^* instead of α, R; thus $\hat{v} \equiv \hat{v}(y; \gamma, R^*)$ according to the definition used in (18).

Figure 5 shows a map of c_i in the (α, R)-plane, such as is obtained for plane Poiseuille flow on the basis of the two-dimensional theory. There is no need here to discuss the differences between this example and similar maps for boundary layers with negative, zero or positive pressure gradient; it seems enough to recognize that, despite these differences which generally arise where R is much greater than the critical value R_c, there is in all cases a distinct range of R above R_c to which the conclusions explained in the next paragraph definitely apply (because, in effect, the different sets of curves all have the same character at the left-hand end of the unstable region). This range at least covers the phase of boundary-layer instability for which the linearized theory is most likely to be useful. Now, the present interest of diagrams like figure 5 is that, because of (22), the values of c_i for three-dimensional waves in the form (21) can also be found from them. Since $\gamma R^* = \alpha R$, these values are found simply by tracing a hyperbola $\alpha R =$ const. to the left from the respective point α on the ordinate for the given R. [For example, if we want c_i for $\alpha = 0{\cdot}2$ and $\beta = 0{\cdot}15$ at $R = 1000$, we have $\gamma = 0{\cdot}25$, $R^* = 800$, and so we find the value of c_i at the point (0·25, 800) on the (α, R)-diagram calculated for two-dimensional waves.]

For $\beta = 0$ a general property illustrated by figure 5 is that, at any given finite R greater than R_c, c_i is positive over a certain finite range of α and has a single maximum in this range. It follows that $\sigma = \alpha c_i$ has a positive maximum, say σ_m, and we write α_m for the respective value of α; thus α_m is determined as a function of R by satisfying the equation

$$\frac{\partial \sigma}{\partial \alpha} = \left[1 + \alpha \frac{\partial}{\partial \alpha}\right] c_i(\alpha, R) = 0. \tag{23}$$

To admit the method of asymptotic approximation used in § 2, the present case must provide the property that, for a given supercritical value of R, σ_m is greater than any value of σ obtained with $\beta \neq 0$, i.e. the optimum two-dimensional wave is the most unstable for the given R. For confirmation on this point, which is actually a rather delicate one having no directly obvious proof, we may refer to the work of Watson (1960). He showed in general that $\sigma(\alpha, \beta, R)$ is greatest for $\alpha = \alpha_m$, $\beta = 0$ at least when R is given in the range above R_c terminated by the value at which c_i has its absolute maximum (e.g. R' in figure 5). For higher Reynolds numbers no definite result can be stated; but in the case of Poiseuille flow Watson assessed the available numerical evidence to imply that the most unstable wave is a two-dimensional one at all Reynolds numbers above the critical.†

† Being based on Squire's theorem these deductions depend crucially on the assumption that the flow is effectively unbounded in z, as also does the paramount deduction that the critical Reynolds number is determined by a two-dimensional wave. But it may be noted incidentally that the mathematical statement of Squire's theorem also holds for complex values of β, in which case the present conclusions are invalidated. In particular, for a disturbance with real exponential z-dependence, β is purely imaginary and we have $\gamma < \alpha$ for $|\beta| < \alpha$, which indicates by virtue of (22) that such a disturbance can be unstable at Reynolds numbers below the critical value for two-dimensional waves. Such disturbances are clearly possible in the presence of a lateral boundary to the flow; more precisely, there can be disturbances which become exponentially decreasing in z outside the extra 'boundary layer' formed in the primary flow. For this reason it may prove impossible to avoid some measure of 'premature' instability in experiments simulating plane Poiseuille flow; because whatever form of lateral boundaries are used, there can be disturbances between the parallel planes which are to some degree sensitive to the presence of these boundaries.

Proceeding from (21), the next step is to introduce a double Fourier integral to represent a localized disturbance. As an extension of the method that was used in §2, we shall here take account of y-variations and so represent a three-dimensional disturbance *in toto*, rather than restricting as before to the distribution over a particular (x, z)-plane. We recognize in advance, however, that this extension scarcely adds to what is directly deducible by the original method which, if applied again, would still afford a complete description of events in any given (x, z)-plane consequent to arbitrary initial conditions in that plane. The following three-dimensional synthesis represents the y-dependence which is apparently concomitant with a (x, y)-distribution initially specified in a chosen plane, and so it does not add to the degree of arbitrariness available in setting an initial-value problem. We shall not attempt to establish whether or not the initial distribution in three dimensions might be represented with greater generality, i.e. to seek a uniqueness theorem supporting the present method of representation. But presumably the method is in fact adequate, particularly as disturbances of the type (22) with harmonic dependence on x and z are apparently the only ones possible in the present theoretical model which is unbounded in x and z, and also as the eigenfunctions $\hat{v}$ presumably may form a complete set. In any case it appears that at least every unstable element of any disturbance is represented here; and so in this context, where the aim is to derive properties due to the class of unstable waves, there would be little point in pursuing the mathematical issues relating to generalization of the initial-value problem.

We accordingly consider

$$v(x,y,z,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(\alpha,\beta)\,\hat{v}(y;\gamma,R^*)\exp\{i(\alpha x+\beta z-\alpha c_r t)+\sigma t\}\,d\alpha\,d\beta, \quad (24)$$

where $\delta(\alpha, \beta)$ is an arbitrary function. If at $t = 0$ the perturbation v is prescribed over a certain plane $y = 0$, say, and if the Fourier transform of $v(x, 0, z, 0)$ is $g(\alpha, \beta)$, then we have that $\delta(\alpha,\beta) = g(\alpha,\beta)/\hat{v}(0;\gamma,R^*)$. Hence, when this expression for $\delta(\alpha, \beta)$ is substituted in the integral, v is determined by (24) for all $t \geqslant 0$.

In order to derive an asymptotic approximation to (24) for large t, the amplification rate σ has to be expanded as far as quadratic terms in $\alpha - \alpha_m$ and β. [For the time being we only consider the maximum at $(\alpha_m, 0)$, leaving the equal one at $(-\alpha_m, 0)$ to be treated later.] The corresponding expansion of the frequency $\omega = \alpha c_r$ is also needed. Considering further the case of Poiseuille flow or boundary layers, we shall now show how the required expansions may be deduced from $\omega + i\sigma = \alpha\mathbf{c}(\gamma, R^*)$, where the function $\mathbf{c} = \mathbf{c}_r + i\mathbf{c}_i$ is in principle known from the solution to the respective two-dimensional problem. Using suffix m to refer always to the optimum conditions $\beta = 0$, $\alpha = \gamma = \alpha_m$, $R^* = R$, we have $(\partial\sigma/\partial\alpha)_m = 0$ by definition (cf. (23)); and we get, using (23) to obtain the second equality,

$$\left(\frac{\partial^2\sigma}{\partial\alpha^2}\right)_m = 2\left(\frac{\partial\mathbf{c}_i}{\partial\gamma}\right)_m + \alpha_m\left(\frac{\partial^2\mathbf{c}_i}{\partial\gamma^2}\right)_m$$

$$= -\frac{2}{\alpha_m}(\mathbf{c}_i)_m + \alpha_m\left(\frac{\partial^2\mathbf{c}_i}{\partial\gamma^2}\right)_m = -2p, \quad \text{say}, \quad (25)$$

where p must be a positive constant since σ_m is a maximum. The first two equalities in (25) indicate how p can be evaluated from data such as are plotted in figure 5. It is next observed that

$$\frac{\partial \sigma}{\partial \beta} = \frac{\alpha\beta}{\gamma}\frac{\partial \mathbf{c}_i}{\partial \gamma} - \frac{\alpha^2\beta R}{\gamma^3}\frac{\partial \mathbf{c}_i}{\partial R^*}, \tag{26}$$

since $$\frac{\partial \gamma}{\partial \beta} = \frac{\beta}{\gamma}, \quad \frac{\partial R^*}{\partial \beta} = \alpha R \frac{\partial}{\partial \beta}\left(\frac{1}{\gamma}\right) = -\frac{\alpha\beta R}{\gamma^3}.$$

Hence we get $$\left(\frac{\partial \sigma}{\partial \beta}\right)_m = 0, \quad \left(\frac{\partial^2 \sigma}{\partial \alpha \partial \beta}\right)_m = 0, \tag{27}$$

and $$\begin{aligned}\left(\frac{\partial^2 \sigma}{\partial \beta^2}\right)_m &= \left(\frac{\partial \mathbf{c}_i}{\partial \gamma}\right)_m - \frac{R}{\alpha_m}\left(\frac{\partial \mathbf{c}_i}{\partial R^*}\right)_m \\ &= -\frac{1}{\alpha_m}\left\{(\mathbf{c}_i)_m + R\left(\frac{\partial \mathbf{c}_i}{\partial R^*}\right)_m\right\} = -2q, \quad \text{say.}\end{aligned} \tag{28}$$

Here q is positive as an implication of Watson's proof, referred to above, that σ is greatest for the optimum two-dimensional wave; and (28) shows how q also is deducible from data such as in figure 5. The required approximation to σ is therefore

$$\sigma = \sigma_m - p(\alpha - \alpha_m)^2 - q\beta^2. \tag{29}$$

Consider next the frequency $\omega = \alpha c_r$. In the same way that (26) leads to (27), we get

$$\left(\frac{\partial \omega}{\partial \beta}\right)_m = 0, \quad \left(\frac{\partial^2 \omega}{\partial \alpha \partial \beta}\right)_m = 0,$$

and so we have

$$\omega = \omega_m + \left(\frac{\partial \omega}{\partial \alpha}\right)_m (\alpha - \alpha_m) + \frac{1}{2}\left(\frac{\partial^2 \omega}{\partial \alpha^2}\right)_m (\alpha - \alpha_m)^2 + \frac{1}{2}\left(\frac{\partial^2 \omega}{\partial \beta^2}\right)_m \beta^2 \tag{30}$$

to the second order in $\alpha - \alpha_m$ and β. Here $\omega_m = \alpha_m(\mathbf{c}_r)_m$, and the three derivatives can be found from the solution to the two-dimensional problem, just as was shown above for the derivatives of σ.

We now introduce

$$X = x - U_m t, \tag{31}$$

where $$U_m = \left(\frac{\partial \omega}{\partial \alpha}\right)_m = (\mathbf{c}_r)_m + \alpha_m\left(\frac{\partial \mathbf{c}_r}{\partial \gamma}\right)_m$$

is the *group velocity* of the wave of maximum amplification, and we also write

$$\left.\begin{aligned} V = U_m - (c_r)_m = \alpha_m\left(\frac{\partial \mathbf{c}_r}{\partial \gamma}\right)_m, \quad \zeta = \alpha - \alpha_m, \\ 2\kappa = -\left(\frac{\partial^2 \omega}{\partial \alpha^2}\right)_m, \quad 2\mu = -\left(\frac{\partial^2 \omega}{\partial \beta^2}\right)_m. \end{aligned}\right\} \tag{32}$$

Hence, when the approximation (30) is substituted, the imaginary part of the exponent in (24) can be arranged in the form

$$\alpha_m(X + Vt) + \zeta X + \beta z + \kappa\zeta^2 t + \mu\beta^2 t. \tag{33}$$

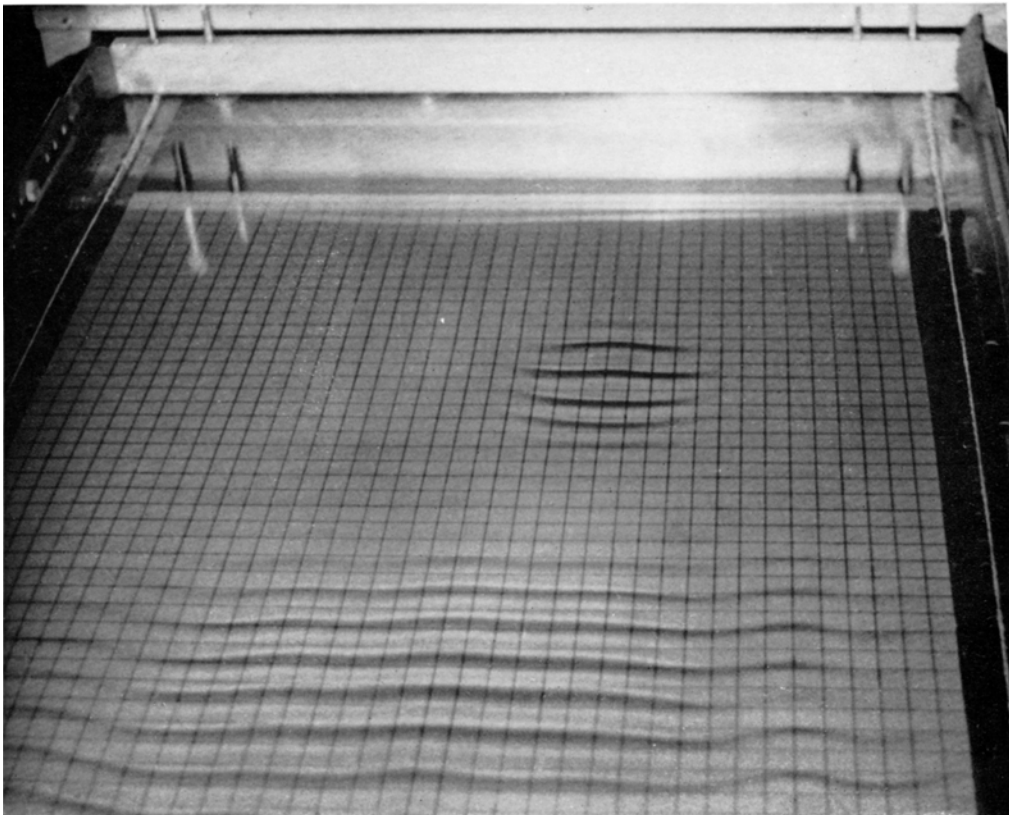

FIGURE 4. Photograph showing an elliptical patch of waves developing on an unstable film of water. The graticule which can be seen here consisted of $\frac{1}{2}$ in. squares and was attached to the under side of the sheet of glass down which the film was running.

In deriving an asymptotic approximation to (24) by Laplace's method, (29) and (33) are the expansions to second order required to account for the maximum of σ at $(\alpha_m, 0)$. The expansions appropriate to the other maximum at $(-\alpha_m, 0)$ are immediately deducible from (29) and (33) on recognition of the fact, noted below (20), that $\mathbf{c}(-\gamma, R^*) = \tilde{\mathbf{c}}(\gamma, R^*)$. In terms of the new variables $\zeta' = -(\alpha + \alpha_m)$, $\beta' = -\beta$, the form of the expansion of σ is seen to be unchanged from (29), while the form of the expansion of ω is seen to be the same as (33) except for an over-all change of sign. Thus, for the two maxima, the respective approximations to the exponential factor in (24) are complex conjugates. In keeping with Laplace's method, the areas of integration of the respective approximations may each be allowed to cover the whole plane; and the method also permits the factor $g(\alpha, \beta)\, \hat{v}(y; \gamma, R^*)/\hat{v}(0; \gamma, R^*)$ in the integrand to be evaluated at the maxima (where $\gamma = \pm\alpha_m$, $R^* = R$) and hence taken outside the integral signs. Use can also be made of the fact that $\hat{v}(y; -\alpha_m, R)$ is the conjugate of $\hat{v}(y; \alpha_m, R)$, which is easily verified from the Orr–Sommerfeld equation. The approximation to (24) is therefore expressible in the form

$$v \sim g(\alpha_m, 0)\, I + g(-\alpha_m, 0)\, \tilde{I}, \tag{34}$$

where

$$I = \left[\frac{\hat{v}(y; \alpha_m, R)}{\hat{v}(0; \alpha_m, R)}\right] \exp\{\sigma_m t + i\alpha_m(X + Vt)\}$$
$$\times \int_{-\infty}^{\infty} \exp\{i\zeta X + (-p + i\kappa)\,\zeta^2 t\}\, d\zeta \int_{-\infty}^{\infty} \exp\{i\beta z + (-q + i\mu)\,\beta^2 t\}\, d\beta, \tag{35}$$

and $\tilde{I}$ is the conjugate of I.

The exponential factor and the integrals in (35) provide the most interesting features of this result, and we may conveniently explain the other features and dismiss them here before performing the integrations. Two of the properties demonstrated might have been expected from the start. First, it is seen that whatever the initial distribution of wave components in the (α, β)-plane, the disturbance depends asymptotically only on the densities of the distribution at the points $(\pm\alpha_m, 0)$, i.e. on the measure of the optimum two-dimensional waves in a Fourier analysis of the initial disturbance. Secondly, the y-distribution of the developing disturbance is shown to become asymptotically the same as that of the optimum wave. Both these properties have been established by the simplest detail of Laplace's general method, namely that some factors in the integrand which are independent of the large parameter t can be evaluated at the maxima of the real exponent σt; the principle is, of course, that the (α, β)-variations of these factors are insignificant in comparison with variations of the real exponential. That the $\hat{v}$ factors can be so treated, whereas variations in the x- and z-dependent factors must be allowed for, derives from the nature of the function $\hat{v}$, or more specifically from the boundary conditions which it must satisfy. For whereas the ultimate range of an unstable disturbance has no prescribed limit in x or z, which accords with x and y entering the integrand in harmonic functions, the range of variation in y is essentially limited. Even for a boundary layer where v is required to vanish on only one plane, the condition that $\hat{v}$ remain bounded for $y \to \infty$ requires that $\hat{v} = e^{-\gamma y}$ outside the layer (see, for instance, Schlichting

1955, p. 320). Thus changes in y-structure cannot continue indefinitely, such as can on the other hand in the 'opening out' of a wave train; and it is easy to see physically that the y-structure 'fixes' when the wave components near the optimum become predominant [e.g. $v \sim e^{-\alpha_m y} \times$ (function of x, z, t) outside a boundary layer].

Evaluation of the integrals in (35) is straightforward. Leaving aside the features explained in the last paragraph, and also dropping a numerical factor which is immaterial here, we can state the general result as follows. On any plane inside the flow and parallel to the boundaries, the distribution of the disturbance is given asymptotically by some linear combination of the real and imaginary parts of

$$\frac{\exp\{\sigma_m t + i\alpha_m(X+Vt)\}}{t}\exp\left\{-\left(\frac{A+iB}{4t}\right)X^2 - \left(\frac{C+iD}{4t}\right)z^2\right\}, \tag{36}$$

where
$$A+iB = \frac{p+i\kappa}{p^2+\kappa^2}, \quad C+iD = \frac{q+i\mu}{q^2+\mu^2}. \tag{37}$$

The point needing emphasis here is that a result of this form is applicable to every problem of the general class in question. Special attention has been paid to Poiseuille flow and boundary layers because of their importance and also to give an example of how the coefficients in (36) may be evaluated; but it is implied that this set of coefficients is a definite attribute of every physical system which can be unstable in the general manner under consideration. (For a film of liquid flowing under gravity, we recall from §2 that $p = a$, $q = b$, $\kappa = 0$, $\mu = 0$; also $V = 0$ so that $X = x'$.)

The following is a list of the properties expressed by this result and stands as a generalization of the set of properties noted below (15) in §2:

(1) The over-all amplitude of the disturbance increases as $t^{-1}e^{\sigma_m t}$.

(2) The disturbance is centred at the origin of (X, z) since the factor $\exp\{-(AX^2+Cz^2)/4t\}$ describes a 'Gaussian' distribution of amplitude along any radius from the origin. The disturbance is therefore effectively confined within an expanding elliptical region outlined by

$$(AX^2+Cz^2)/4t = \text{const.}, \quad \text{say about 3.}$$

The axes of the ellipse both increase as $t^{\frac{1}{2}}$, and consequently its area increases linearly with t.

(There is apparently no general conclusion to be drawn as regards the relative magnitudes of A and C and hence the eccentricity of the ellipse. We recall from §2 that when a film is made marginally unstable, then $C/A = a/b \simeq 0$ and so the ellipse is greatly elongated in the z-direction; but this feature is peculiar to the problem and does not arise in general.)

(3) Since $x = U_m t$ at $X = 0$, the disturbance is carried downstream at the group velocity U_m associated with the two-dimensional wave of maximum amplification.

(4) The (X, z)-distribution of the disturbance is modulated by this wave travelling at velocity $-V$ relative to X; i.e. the wave travels downstream at its phase velocity $(c_r)_m = U_m - V$, but is cancelled everywhere except in the elliptical patch which advances through the wave train at relative velocity V.

(5) Another effect of dispersion due to variable c_r is represented by the factor $\exp\{-i(BX^2+Dz^2)/4t\}$. The disturbance is thus modulated by undulations whose spacing rapidly decreases with distance from the centre; however, they will only appear as a prominent feature of the disturbance if B and D are considerably greater than A and C respectively. For these undulations the contours of constant phase are ellipses whose areas increase linearly with time, and the pattern formed by superposition of them upon the Gaussian distribution considered in (2) above is similar at all times.

Finally, the points labelled (*b*) and (*c*) at the end of §2 may be recognized to apply generally to the foregoing results. The matter (*c*) regarding limitations on the theory due to non-linear effects particularly deserves re-emphasis, since these limitations are liable to be more severe for other problems than for the one investigated in §2. For instance, disturbances in unstable boundary layers are known to develop dependence on non-linear interactions very readily unless under conditions where their initial magnitude is unusually small, and the extent to which the properties described here may be actually manifested remains in doubt in the absence of any direct evidence. Nevertheless, the reasons given in §1 seem definitely to suggest that present ideas may often be relevant to a certain incipient phase of instability, in plane Poiseuille flow and boundary layers as well as the incontestable example demonstrated in §§2 and 3, and these ideas may generally provide a rather more realistic account of events in this phase than the basic two-dimensional form of stability theory.

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